SEMESTER TWO

MATHEMATICS
SPECIALIST
UNITS 3 & 4

2016

SOLUTIONS

Calculator–free Solutions

1. $x = 0 \quad \rightarrow \quad y = \pm\frac{\sqrt{3}}{2}$ ✓

$2x + 2 + 8y\frac{dy}{dx} = 0$ ✓✓

$\therefore \frac{dy}{dx} = -\frac{x+1}{4y}$

$\left.\frac{dy}{dx}\right|_{0,-\frac{\sqrt{3}}{2}} = -\frac{1}{\left(4 \times \left(-\frac{\sqrt{3}}{2}\right)\right)} = \frac{1}{2\sqrt{3}}$ ✓

$\therefore\ y = \frac{x}{2\sqrt{3}} - \frac{\sqrt{3}}{2}$ ✓ [5]

2. (a) $(1+i)^5 + (1-i)^5 = \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^5 + \left(\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^5$ ✓

$= 4\sqrt{2}\,\text{cis}\left(\frac{5\pi}{4}\right) + 4\sqrt{2}\,\text{cis}\left(-\frac{5\pi}{4}\right)$ ✓

$= 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) - i\sin\left(\frac{5\pi}{4}\right)\right]$ ✓

$= 4\sqrt{2}\left[2\cos\left(\frac{5\pi}{4}\right)\right]$

$= 8\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -8$ ✓✓

(b) (i) $2\,Re(z) + 3\,Im(z) \geq 3$ ✓✓

(ii) $-\frac{\pi}{2} \leq Arg(z) < \frac{\pi}{3}$ and $|z| \leq 3$ ✓✓✓ [10]

3. (a) $\int \cos x\,(1 - \cos x)\,dx = \int \cos x\ dx - \int \cos^2 x\ dx$

$= \sin x - \int \frac{1 + \cos(2x)}{2}\,dx$ ✓✓

$= \sin x - \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\,dx$

$= \sin x - \frac{x}{2} - \frac{1}{4}\sin(2x) + C$ ✓✓

3. (b) Let $u = 1 + \cos(2x)$ ✓

Then $\dfrac{du}{dx} = -2\sin(2x)$ and hence $-\dfrac{du}{2} = \sin(2x)\,dx$ ✓

Also, $u\left(\dfrac{\pi}{6}\right) = 1 + \cos\left(\dfrac{\pi}{3}\right) = 1 + \dfrac{1}{2} = \dfrac{3}{2}$

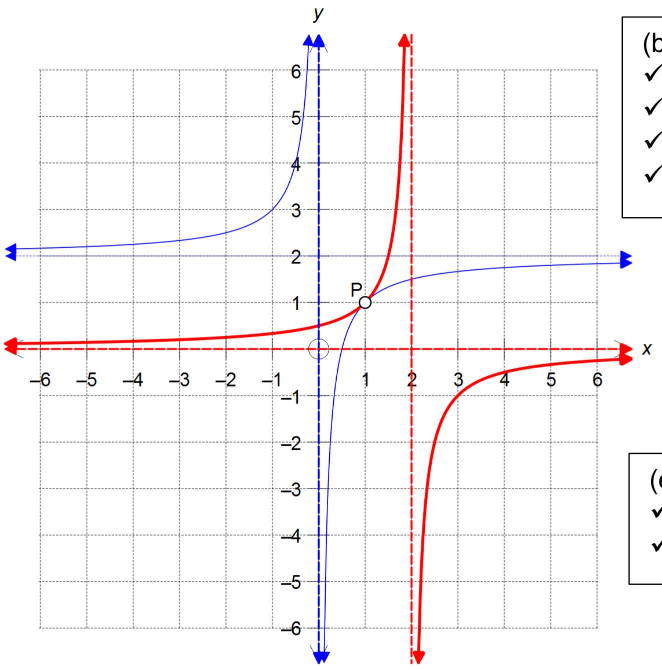
and $u\left(\dfrac{\pi}{4}\right) = 1 + \cos\left(\dfrac{\pi}{2}\right) = 1$ ✓

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin(2x)}{1+\cos(2x)}\,dx = \int_{\frac{3}{2}}^{1} \frac{-\frac{du}{2}}{u} = \frac{1}{2}\int_{1}^{\frac{3}{2}} \frac{du}{u}$$ ✓

$$= \frac{1}{2}\Big[\ln|u|\Big]_{1}^{\frac{3}{2}} = \frac{1}{2}\ln\left(\frac{3}{2}\right)$$ ✓ [9]

4. (a) $f(x) = 2 + \dfrac{1-x}{x^2-x}$ $\left(= 2 - \dfrac{1}{x}\right)$ ✓✓✓

(b) (e)



(b)
✓ Reciprocal Curve
✓ Marks undefined point P
✓ Vertical asymptote at x=0
✓ Horizontal asymptote at y=2

(e) (in bold)
✓ Asymptotes x=2 and y=0
✓ Marks undefined point P

(c) Range = $\{y \in R : y \neq 1 \text{ and } y \neq 2\}$ ✓

(d) $y = 2 - \dfrac{1}{x}$ hence $\dfrac{1}{x} = 2 - y$ and $x = \dfrac{1}{2-y}$ ✓

therefore $f^{-1}(x) = \dfrac{1}{2-x}$ as required

domain = $\{x \in R : x \neq 2 \text{ and } x \neq 1\}$ ✓ [12]

5. (a) $\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx = \left[\sin x + \cos x \right]_0^{\frac{\pi}{4}}$ ✓

$= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \sin(0) - \cos(0) = \sqrt{2} - 1 \text{ units}^2$ ✓

(b) $\pi\int_0^{\frac{\pi}{4}} \cos^2 x \; dx \; - \; \pi\int_0^{\frac{\pi}{4}} \sin^2 x \; dx$ ✓

$= \pi\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos(2x) - \frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$ ✓

$= \pi\int_0^{\frac{\pi}{4}} \cos(2x)\, dx$ ✓

$= \pi\left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}$ ✓

$= \frac{\pi}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] = \frac{\pi}{2} \text{ units}^3$ ✓ [7]

6. (a) $\left[\begin{array}{cccc} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & q-2 \\ 0 & 0 & 1-2p & q+24 \end{array}\right]$ $R_2 + 2R_3$ ✓✓

(b) $\left[\begin{array}{cccc} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -9 & 27 \end{array}\right]$

$-9z = 27 \quad \therefore z = -3$ ✓

$2y - 3 = 1 \quad \therefore y = 2$ ✓

$2x + 2 - 3 = 1 \quad \therefore x = 1$ ✓

(c) (i) $p = \frac{1}{2}$ and $q = -24$ ✓

(ii) $p = \frac{1}{2}$ and $q \neq -24$ ✓ [7]

Calculator–assumed Solutions

7. (a) $z = 1 - 2i$ and $z = -3i$ ✓✓

(b) $Q(z) = (z - 1 - 2i)(z - 1 + 2i)(z - 3i)(z + 3i)$ ✓

(c) CAS → $Q(z) = z^4 - 2z^3 + 14z^2 - 18z + 45$

$\therefore\ a = -2,\ b = 14,\ c = -18$ and $d = 45$ ✓✓ [5]

8. (a) $f(g(x)) = 5 + \sqrt{g} = 5 + \sqrt{(x-5)^2} = 5 + |x - 5|$ ✓

Domain = $\{x: x \in \mathbb{R}\}$ ✓

Range = $\{y: y \geq 5\}$ ✓

$g(f(x)) = (f - 5)^2 = (5 + \sqrt{x} - 5)^2 = (\sqrt{x})^2 = x$ ✓

Domain = $\{x: x \geq 0\}$ ✓

Range = $\{y: y \geq 0\}$ ✓

(b) $f(g(a) = g(f(a))\quad \therefore f\left((a-5)^2\right) = g(5 + \sqrt{a})$

$5 + \sqrt{(a-5)^2} = (5 + \sqrt{a} - 5)^2$ ✓

$5 + |a - 5| = a$

$|a - 5| = a - 5$

$\therefore a \geq 5$ ✓

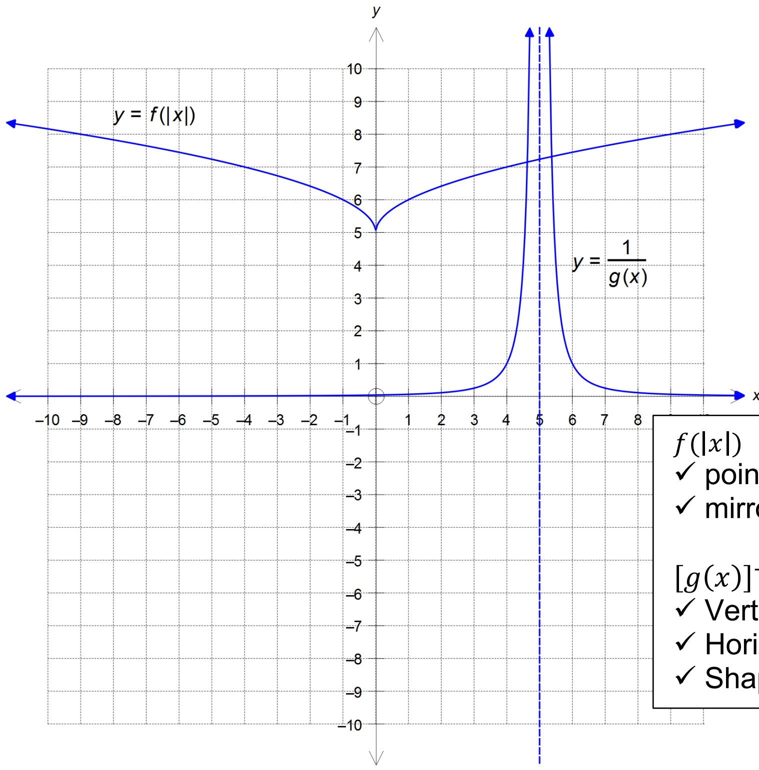
(c) $g(h) = x^2 + 6x + 9$

$= (x + 3)^2 = (h - 5)^2$ ✓

$\therefore h - 5 = \pm(x + 3)$ hence ✓

$h(x) = x + 8$ or $h(x) = -x + 2$

(d)



$f(|x|)$
✓ point of reflection
✓ mirror shape on 2nd quadrant

$[g(x)]^{-1}$
✓ Vertical asymptote at x=5
✓ Horizontal asymptote at y=0
✓ Shape and accuracy

[15]

9. (a) $\dfrac{dP}{dt} = kP$ hence $\dfrac{dP}{P} = kdt$

$\therefore \displaystyle\int \frac{dP}{P} = \int kdt$ ✓

$\ln|P| = kt + C$ ✓

$\therefore P = e^{kt + C} = e^{kt} \times e^{C} = P_0\, e^{kt}$ ✓

(b) (i) At $t = 1$, $P(1) = P_0\, e^k$ thus $\dfrac{P}{P_0} = e^k = 1.24$ ✓

$\therefore k = \ln(1.24) = 0.2151$ ✓

therefore, $P(t) = 2500\, e^{0.2151t}$

and $P(2.5) = 2500\, e^{2.5 \times 0.2151} = 4280.38$

~ 4280 fish ✓

(ii) $2500e^{0.2151t} = 20000$ ✓

$t = \dfrac{\ln(8)}{0.2151} = 9.67$ $\therefore$ 10 years ✓ [8]

10. (a) Point of reference is P(5, 0). From P:

$x = 2\cos(4t)$

$\therefore \dfrac{dx}{dt} = -8\sin(4t)$ ✓

and $\dfrac{d^2x}{dt^2} = -32\cos(4t) = -16(2\cos(4t))$ ✓

$\therefore \dfrac{d^2x}{dt^2} = -4^2 x$ which is the condition for SHM ✓

(b) $\dfrac{dx}{dt} = |-8\sin(4t)| = 4$

$\therefore |\sin(4t)| = \pm\dfrac{1}{2}$ and hence $4t = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$, etc ✓

and $t = \dfrac{\pi}{24}, \dfrac{5\pi}{24}, \dfrac{7\pi}{24}, \dfrac{11\pi}{24}, \dfrac{13\pi}{24}, \dfrac{17\pi}{24}$, etc

$x\left(\dfrac{\pi}{24}\right) = 5 + 2\cos\left(\dfrac{\pi}{6}\right) = 5 + \sqrt{3}$ $\therefore (5 + \sqrt{3}, 0)$ ✓

$x\left(\dfrac{5\pi}{24}\right) = 5 + 2\cos\left(\dfrac{5\pi}{6}\right) = 5 - \sqrt{3}$ $\therefore (5 - \sqrt{3}, 0)$ ✓

(c) $x = -1 = 2\cos(4t)$

$\therefore \ddot{x} = -16(x) = -16(-1) = 16$ $\therefore \pm 16\ ms^{-2}$ ✓ [7]

11. (a) (i) $2\sqrt{3} - 2i = 4 \text{ cis}\left(-\frac{\pi}{6}\right)$ ✓

Other roots are $\pm\frac{2\pi}{3}$ apart, hence

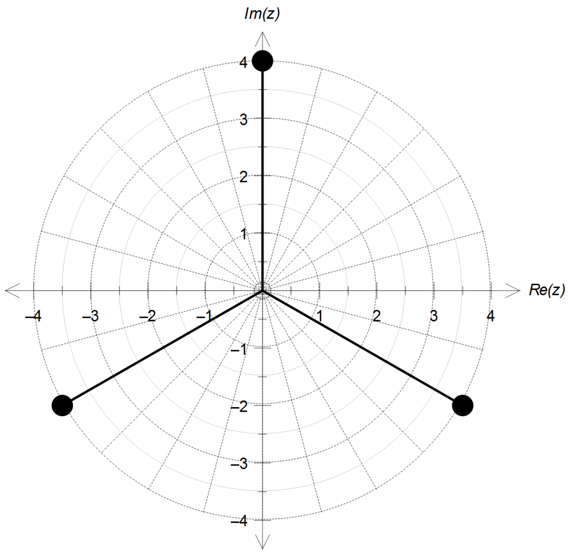
$z = 4 \text{ cis}\left(-\frac{\pi}{6} + \frac{2\pi}{3}\right) = 4 \text{ cis}\left(\frac{\pi}{2}\right)$ ✓

$z = 4 \text{ cis}\left(-\frac{\pi}{6} - \frac{2\pi}{3}\right) = 4 \text{ cis}\left(-\frac{5\pi}{6}\right)$ ✓

(ii) $4 \text{ cis}\left(\frac{\pi}{2}\right) = 4i$ ✓

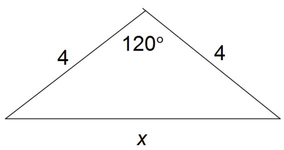
$4 \text{ cis}\left(-\frac{5\pi}{6}\right) = -2\sqrt{3} - 2i$ ✓

(b)



✓ Same magnitude
✓ Equally spaced 120° apart

(c)



$x^2 = 16 + 16 - 2(16)\cos(120°)$ ✓

$x^2 = 32 - 32\left(-\frac{1}{2}\right) = 48$

$x = 4\sqrt{3}$ ✓

$\therefore$ perimeter $= 12\sqrt{3}$ units ✓ [10]

12. (a) $4 = 1.96\left(\dfrac{12}{\sqrt{n}}\right)$ ✓✓

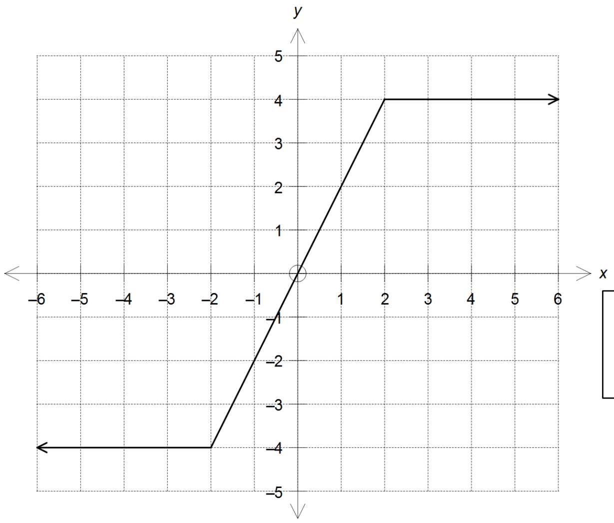
$\therefore n = 34.57$ i.e. 35 calculators are needed ✓

(b) $94 \pm 1.645\left(\dfrac{12}{\sqrt{60}}\right)$ ✓✓

$\therefore (91.45 , 96.55)$ ✓

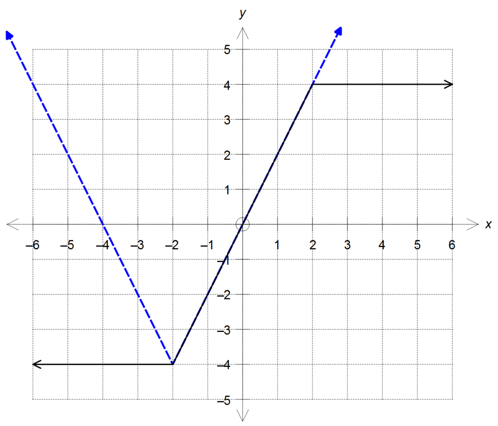
(c) No, as the 90% interval does not contain the 100hr claimed. ✓✓ [8]

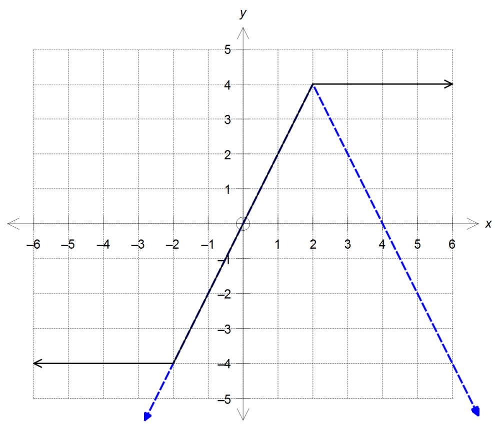
13. (a)



✓ line y = 2x
✓ lines y = 2 and y = –2

(b) Two possible solutions:





hence, $a = 2$, $b = 2$ and $c = –4$ ✓✓✓

or $a = –2$, $b = –2$ and $c = 4$ ✓ [6]

14. (a) $\begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix}$ ✓

$r = \begin{pmatrix}4\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$ ✓

(b) $\begin{pmatrix}4+2\lambda\\-2-\lambda\\3+3\lambda\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\3\end{pmatrix} = 5$ ✓

CAS → $\lambda = -1$ ✓

$\therefore \begin{pmatrix}4-2\\-2+1\\3-3\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}$ ✓

(c) $\begin{pmatrix}4\\-2\\3\end{pmatrix} - \begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix}$ ✓

$\therefore \left|\begin{matrix}2\\-1\\3\end{matrix}\right| = \sqrt{14}$ units ✓

(d) Use the normal to the xy plane and the normal to the plane π : ✓

$\begin{pmatrix}2\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = \left|\begin{matrix}2\\-1\\3\end{matrix}\right| \times \left|\begin{matrix}0\\0\\1\end{matrix}\right| \times \cos\theta$ ✓

$\therefore \theta = 36.7°$ ✓

(e) Create a line perpendicular to the plane that passes through O:

Let $r = \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$

Find the point of intersection of this line and the plane:

$\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\3\end{pmatrix} = 5$ ✓

CAS → $\lambda = \frac{5}{14}$ ✓

Hence, $\left|\frac{5}{14}\begin{pmatrix}2\\-1\\3\end{pmatrix}\right| = \frac{5}{14}\sqrt{14}$, so equation is $|\boldsymbol{r}| = \frac{5}{14}\sqrt{14}$ ✓

And the point of tangency is:

$r = \frac{5}{14}\begin{pmatrix}2\\-1\\3\end{pmatrix}$ ✓ [14]

15. (a) $\dfrac{\delta V}{\delta r} \approx \dfrac{dV}{dr}$

$\therefore \delta V \approx \delta r \dfrac{dV}{dr} = \delta r \times 4kr^3$ ✓

$\therefore \dfrac{\delta V}{V} \approx \dfrac{\delta r \times 4kr^3}{kr^4} = 4 \times \dfrac{\delta r}{r}$ ✓

$\therefore \dfrac{\delta r}{r} \approx \dfrac{1}{4}\dfrac{\delta V}{v} = \dfrac{1}{4} \times 10\% = 2.5\%$ ✓

(b) rate given: $\dfrac{dr}{dl} = -0.05 \dfrac{\boldsymbol{mm}}{\boldsymbol{cm}}$

rate needed: $\dfrac{dV}{dl}$

hence, $\dfrac{dV}{dl} = \dfrac{dV}{dr} \times \dfrac{1}{\frac{dl}{dr}}$ ✓

$= \left(4 \times 15 \times r^3\right) \times -0.05 = -3r^3$ ✓

$r = 2.2 - 12 \times 0.05 = 1.6$ ***mm*** ✓

$\therefore \dfrac{dV}{dl} = -3 \times 1.6^3 = -12.288 \dfrac{\boldsymbol{mm}^3}{\boldsymbol{cm}}$ ✓ [7]

16. (a) $C\left(42.5 - 1.96 \times \dfrac{8}{\sqrt{32}} \le \mu \le 42.5 + 1.96 \times \dfrac{8}{\sqrt{32}}\right) = 0.95$ ✓

$C(39.73 \le \mu \le 45.27) = 0.95$ ✓✓

The engineers can be 95% confident that the mean operating temperature of the lithium-ion batteries is somewhere between 39.73°C and 45.27°C. ✓

(b) $n = \dfrac{(2.576)^2 \times 8^2}{(0.5)^2} = 1698.76$ ✓✓

They will need 1699 batteries to be within 0.5°C of the true mean. ✓ [7]

17. (a) $x(t) = \left(\cos t - \frac{1}{2} \right) = 0$

$\therefore \cos t = \frac{1}{2}$ hence $t = \frac{\pi}{3}$ ✓

$\mathbf{v}(t) = \frac{dx}{dt} = (-\sin t)\,\mathbf{i} + (-\cos t)\,\mathbf{j}$ ✓

$\therefore \left| \mathbf{v}\left(\frac{\pi}{3}\right) \right| = \left| \left(-\sin\left(\frac{\pi}{3}\right) \right) \mathbf{i} + \left(-\cos\left(\frac{\pi}{3}\right) \right) \mathbf{j} \right|$ ✓

$= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$ units/s ✓

(b) $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = (-\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$ ✓

$\therefore |\mathbf{a}(t)| = \sqrt{(-\cos t)^2 + (\sin t)^2} = \sqrt{1} = 1$ unit/s^2 ✓

(c) $\sin t = 2 - y$ and $\cos t = x + \frac{1}{2}$

since $\sin^2 t + \cos^2 t = 1$

then $\left(x + \frac{1}{2} \right)^2 + (y-2)^2 = 1$ ✓

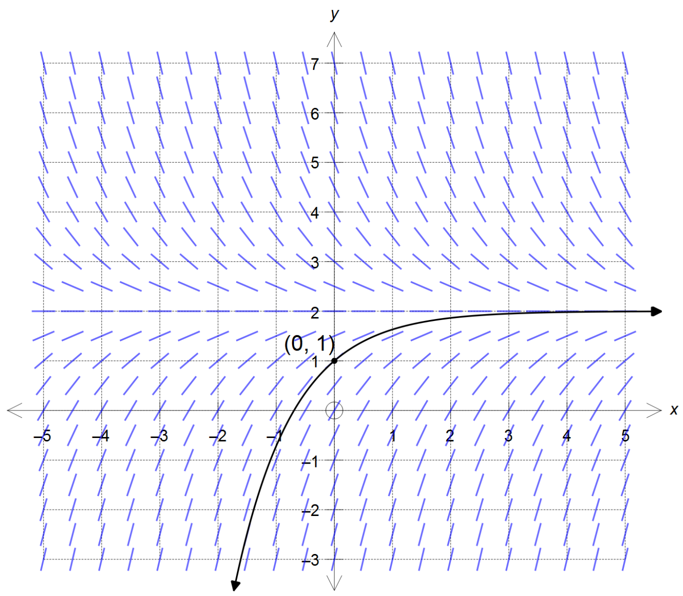
the path is a circle centred at $\left(-\frac{1}{2}, 2 \right)$ of radius = 1 unit ✓ [8]

18. (a) $\frac{dy}{dx} = -y + 2$ ✓✓

Reasons: (at least 1) ✓

- when $\frac{dy}{dx} = 0$ the isoclines follow the line $y = 2$
- the isoclines describe exponential solutions
- the isoclines follow exponential decay, hence "-y"

(b)



✓ exponential curve that follows the isoclines
✓ passes through the point (0, 1)

[5]